Calculus Made Easy

Differential Calculus

&

Integral Calculus

by

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PROLOGUE.

CONSIDERING how many fools can calculate, it is surprising that it should be thought either a difficult or a tedious task for any other fool to learn how to master the same tricks.

Some calculus-tricks are quite easy. Some are enormously difficult. The fools who write the text-books of advanced mathematics—and they are mostly clever fools—seldom take the trouble to show you how easy the easy calculations are. On the contrary, they seem to desire to impress you with their tremendous cleverness by going about it in the most difficult way.

Being myself a remarkably stupid fellow, I have had to unteach myself the difficulties, and now beg to present to my fellow fools the parts that are not hard. Master these thoroughly, and the rest will follow. What one fool can do, another can.

CHAPTER I.

TO DELIVER YOU FROM THE PRELIMINARY TERRORS.

THE preliminary terror, which chokes off most fifth-form boys from even attempting to learn how to calculate, can be abolished once for all by simply stating what is the meaning—in common-sense terms—of the two principal symbols that are used in calculating.

These dreadful symbols are:

(1) d which merely means "a little bit of."

Thus dx means a little bit of x; or du means a little bit of u. Ordinary mathematicians think it more polite to say "an element of," instead of "a little bit of." Just as you please. But you will find that these little bits (or elements) may be considered to be indefinitely small.

(2) $\int$ which is merely a long S, and may be called (if you like) "the sum of."

Thus $\int dx$ means the sum of all the little bits of x; or $\int dt$ means the sum of all the little bits of t. Ordinary mathematicians call this symbol "the integral of." Now any fool can see that if x is considered as made up of a lot of little bits, each of which is called dx, if you add them all up together you get the sum of all the dx's, (which is the same thing as the whole of x). The word "integral" simply means "the whole." If you think of the duration of time for one hour, you may (if you like) think of it as cut up into 3600 little bits called seconds. The whole of the 3600 little bits added up together make one hour.

When you see an expression that begins with this terrifying symbol, you will henceforth know that it is put there merely to give you instructions that you are now to perform the operation (if you can) of totalling up all the little bits that are indicated by the symbols that follow.

That's all.

CHAPTER II.

ON DIFFERENT DEGREES OF SMALLNESS.

WE shall find that in our processes of calculation we have to deal with small quantities of various degrees of smallness.

We shall have also to learn under what circumstances we may consider small quantities to be so minute that we may omit them from consideration. Everything depends upon relative minuteness.

Before we fix any rules let us think of some familiar cases. There are 60 minutes in the hour, 24 hours in the day, 7 days in the week. There are therefore 1440 minutes in the day and 10080 minutes in the week.

Obviously 1 minute is a very small quantity of time compared with a whole week. Indeed, our forefathers considered it small as compared with an hour, and called it "one minùte," meaning a minute fraction—namely one sixtieth—of an hour. When they came to require still smaller subdivisions of time, they divided each minute into 60 still smaller parts, which, in Queen Elizabeth's days, they called "second minùtes" (*i.e.*, small quantities of the second order of minuteness). Nowadays we call these small quantities of the second order of smallness "seconds." But few people know *why* they are so called.

Now if one minute is so small as compared with a whole day, how much smaller by comparison is one second!

Again, think of a farthing as compared with a sovereign: it is worth only a little more than $\frac{1}{1000}$ part. A farthing more or less is of precious little importance compared with a sovereign: it may certainly be regarded as a *small* quantity. But compare a farthing with £1000: relatively to this greater sum, the farthing is of no more importance than $\frac{1}{1000}$ of a farthing would be to a sovereign. Even a golden sovereign is relatively a negligible quantity in the wealth of a millionaire.

Now if we fix upon any numerical fraction as constituting the proportion which for any purpose we call relatively small, we can easily state other fractions of a higher degree of smallness. Thus if, for the purpose of time, $\frac{1}{60}$ be called a *small* fraction, then $\frac{1}{60}$ of $\frac{1}{60}$ (being a *small* fraction of a *small* fraction) may be regarded as a *small quantity of the second order* of smallness.*

Or, if for any purpose we were to take 1 per cent. (*i.e.*, $\frac{1}{100}$) as a *small* fraction, then 1 per cent. of 1 per cent. (*i.e.*, $\frac{1}{10,000}$) would be a small fraction of the second order of smallness; and $\frac{1}{1,000,000}$ would

* The mathematicians talk about the second order of "magnitude" (*i.e.* greatness) when they really mean second order of *smallness*. This is very confusing to beginners.

be a small fraction of the third order of smallness, being 1 per cent. of 1 per cent. of 1 per cent.

Lastly, suppose that for some very precise purpose we should regard $\frac{1}{1,000,000}$ as "small." Thus, if a first-rate chronometer is not to lose or gain more than half a minute in a year, it must keep time with an accuracy of 1 part in 1,051,200. Now if, for such a purpose, we regard $\frac{1}{1,000,000}$ (or one millionth) as a small quantity, then $\frac{1}{1,000,000}$ of $\frac{1}{1,000,000}$, that is, $\frac{1}{1,000,000,000,000}$ (or one billionth) will be a small quantity of the second order of smallness, and may be utterly disregarded, by comparison.

Then we see that the smaller a small quantity itself is, the more negligible does the corresponding small quantity of the second order become. Hence we know that *in all cases we are justified in neglecting the small quantities of the second—or third* (or higher)—*orders*, if only we take the small quantity of the first order small enough in itself.

But it must be remembered that small quantities, if they occur in our expressions as factors multiplied by some other factor, may become important if the other factor is itself large. Even a farthing becomes important if only it is multiplied by a few hundred.

Now in the calculus we write dx for a little bit of x. These things such as dx, and du, and dy, are called "differentials," the differential of x, or of u, or of y, as the case may be. [You *read* them as *dee-eks*, or *dee-you*, or *dee-wy*.] If dx be a small bit of x, and relatively small of itself, it does not follow that such quantities as $x \cdot dx$, or $x^2 dx$, or $a^x dx$ are negligible. But $dx \times dx$ would be negligible, being a small quantity of the second order.

A very simple example will serve as illustration.

Let us think of x as a quantity that can grow by a small amount so as to become $x+dx$, where dx is the small increment added by growth. The square of this is $x^2+2x \cdot dx+(dx)^2$. The second term is not negligible because it is a first-order quantity; while the third term is of the second order of smallness, being a bit of a bit of x. Thus if we took dx to mean numerically, say, $\frac{1}{60}$ of x, then the second term would be $\frac{2}{60}$ of x^2, whereas the third term would be $\frac{1}{3600}$ of x^2. This last term is clearly less important than the second. But if we go further and take dx to mean only $\frac{1}{1000}$ of x, then the second term will be $\frac{2}{1000}$ of x^2, while the third term will be only $\frac{1}{1,000,000}$ of x^2.

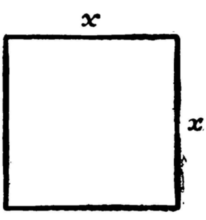

FIG. 1.

Geometrically this may be depicted as follows: Draw a square (Fig. 1) the side of which we will take to represent x. Now suppose the square to grow by having a bit dx added to its size each way. The enlarged square is made up of the original square x^2, the two rectangles at the top and on the right, each of which is of area $x \cdot dx$ (or together $2x \cdot dx$), and the little square at the top right-hand corner which is $(dx)^2$. In Fig. 2 we have taken dx as

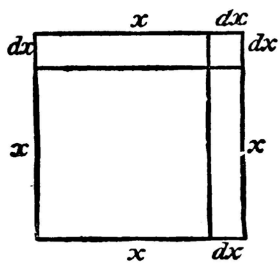

FIG. 2.

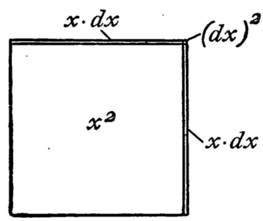

FIG. 3.

quite a big fraction of x—about $\frac{1}{5}$. But suppose we had taken it only $\frac{1}{100}$—about the thickness of an inked line drawn with a fine pen. Then the little corner square will have an area of only $\frac{1}{10,000}$ of x^2, and be practically invisible. Clearly $(dx)^2$ is negligible if only we consider the increment dx to be itself small enough.

Let us consider a simile.

Suppose a millionaire were to say to his secretary: next week I will give you a small fraction of any money that comes in to me. Suppose that the secretary were to say to his boy: I will give you a small fraction of what I get. Suppose the fraction in each case to be $\frac{1}{100}$ part. Now if Mr. Millionaire received during the next week £1000, the secretary would receive £10 and the boy 2 shillings. Ten pounds would be a small quantity compared with £1000; but two shillings is a small small quantity indeed, of a very secondary order. But what would be the disproportion if the fraction, instead of being $\frac{1}{100}$, had been settled at $\frac{1}{1000}$ part? Then, while Mr. Millionaire got his £1000, Mr. Secretary would get only £1, and the boy less than one farthing!

The witty Dean Swift* once wrote:

> "So, Nat'ralists observe, a Flea
> "Hath smaller Fleas that on him prey.
> "And these have smaller Fleas to bite 'em,
> "And so proceed *ad infinitum.*"

An ox might worry about a flea of ordinary size—a small creature of the first order of smallness. But he would probably not trouble himself about a flea's flea; being of the second order of smallness, it would be negligible. Even a gross of fleas' fleas would not be of much account to the ox.

* *On Poetry: a Rhapsody* (p. 20), printed 1733—usually misquoted.

CHAPTER III.

ON RELATIVE GROWINGS.

All through the calculus we are dealing with quantities that are growing, and with rates of growth. We classify all quantities into two classes: *constants* and *variables*. Those which we regard as of fixed value, and call *constants*, we generally denote algebraically by letters from the beginning of the alphabet, such as a, b, or c; while those which we consider as capable of growing, or (as mathematicians say) of "varying," we denote by letters from the end of the alphabet, such as x, y, z, u, v, w, or sometimes t.

Moreover, we are usually dealing with more than one variable at once, and thinking of the way in which one variable depends on the other: for instance, we think of the way in which the height reached by a projectile depends on the time of attaining that height. Or, we are asked to consider a rectangle of given area, and to enquire how any increase in the length of it will compel a corresponding decrease in the breadth of it. Or, we think of the way in which any variation in the slope of a ladder will cause the height that it reaches, to vary.

Suppose we have got two such variables that